7.4 Derivative, integral, and multiplication of Laplace Transforms

we know: $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$$

today: deriv. and integral in s-domain

$$F(s) = \int_0^\infty f(t)\,e^{-st}\,dt$$

$$F'(s) = \frac{d}{ds}\int_0^\infty f(t)\,e^{-st}\,dt$$

$$= \int_0^\infty f(t)\left(\frac{d}{ds}e^{-st}\right)dt = \int_0^\infty f(t)\left(-t\cdot e^{-st}\right)dt$$

$$= \int_0^\infty \left[-t\cdot f(t)\right]e^{-st}\,dt$$

$$\boxed{F'(s) = \mathcal{L}\{-t\cdot f(t)\}}$$

repeat: $F''(s) = \mathcal{L}\{t^2 f(t)\}$ each differentiation gets a factor of $-t$

$$\boxed{F^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\}}$$

useful when finding Laplace transform of t times something on the table

for example, $\mathcal{L}\{t\cosh(6t)\}$ is not on the table but $\mathcal{L}\{\cosh(6t)\}$ is

$$\mathcal{L}\{-t f(t)\} = F'(s)$$

$$\mathcal{L}\{t\cosh(6t)\} = \mathcal{L}\{-t\underbrace{(-\cosh(6t))}_{f(t)}\} = F'(s)$$

find $F(s) = -\dfrac{s}{s^2-36}$

$$\frac{d}{ds}\left(-\frac{s}{s^2-36}\right) = \cdots = \boxed{\frac{s^2+36}{(s^2-36)^2} = \mathcal{L}\{t\cosh(6t)\}}$$

we can also use it the other way

$$F'(s) = \mathcal{L}\{-t\, f(t)\} \qquad \frac{\mathcal{L}^{-1}\{F'(s)\}}{-t} = f(t)$$

for example, $\cancel{\mathcal{L}^{-1}\{}$ find inverse of $\underbrace{\ln\left(\frac{1}{s^2-16}\right)}_{\text{not on table}}$

rewrite: $\ln\left(\frac{1}{s^2-16}\right) = \ln(1) - \ln(s^2-16)$

$$= 0 - \ln[(s+4)(s-4)]$$

$$F = -\ln(s+4) - \ln(s-4)$$

$$F' = -\frac{1}{s+4} - \frac{1}{s-4}$$

$$\mathcal{L}^{-1}\left\{-\frac{1}{s+4} - \frac{1}{s-4}\right\} = -e^{-4t} - e^{4t}$$

then $f(t) = \dfrac{\mathcal{L}^{-1}\{F'\}}{-t} = \boxed{\dfrac{e^{-4t} + e^{4t}}{t}}$

now integral in s-domain, specifically $\int_s^\infty F(\sigma)d\sigma$

start with $F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$

$$\int_s^\infty F(\sigma)\,d\sigma = \int_s^\infty \int_0^\infty f(t)\, e^{-\sigma t}\, dt\, d\sigma$$

swap integration order

$$\int_0^\infty \int_s^\infty f(t)\, e^{-\sigma t}\, d\sigma\, dt$$

$$= \int_0^\infty f(t) \cdot \frac{1}{-t} e^{-\sigma t} \Big|_{\sigma = s}^{\sigma = \infty} dt$$

$$= \int_0^\infty -\frac{1}{t} f(t) \left(0 - e^{-st}\right) dt = \int_0^\infty \left[\frac{f(t)}{t}\right] e^{-st}\, dt$$

$$\boxed{\int_s^\infty F(\sigma)\,d\sigma = \mathcal{L}\left\{\frac{f(t)}{t}\right\}}$$

$\lim\limits_{t \to 0^+} \frac{f(t)}{t}$ must exist useful to transform something over t

$\mathcal{L}\{\text{something}\}$ is known

$f(t) \to F(s)$

$$\mathcal{L}\left\{\frac{1-\cos(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$$

$$\lim_{t\to 0^+} \frac{1-\cos(t)}{t} \text{ must exist}$$

$$= \lim_{t\to 0^+} \frac{\sin(t)}{1} = 0 \text{ (exists)}$$

$$\mathcal{L}\{1-\cos(t)\} = \frac{1}{s} - \frac{s}{s^2+1} = F(s)$$

$$\int_s^\infty \left(\frac{1}{\sigma} - \frac{\sigma}{\sigma^2+1}\right)d\sigma = \ln\sigma - \frac{1}{2}\ln(\sigma^2+1)\Big|_{\sigma=s}^{\sigma=\infty}$$

$$= \ln\sigma - \ln\sqrt{\sigma^2+1}\Big|_{\sigma=s}^{\sigma=\infty}$$

$$= \ln\left(\frac{\sigma}{\sqrt{\sigma^2+1}}\right)\Big|_{\sigma=s}^{\sigma=\infty}$$

$$= \ln(1) - \ln\left(\frac{s}{\sqrt{s^2+1}}\right)$$

$$\boxed{\mathcal{L}\left\{\frac{1-\cos(t)}{t}\right\} = -\ln\left(\frac{s}{\sqrt{s^2+1}}\right)}$$

revisit $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

$$= \int_0^\infty f'(t)e^{-st}dt$$

$$\lim_{s\to 0}\left[sF(s) - f(0)\right] = \lim_{s\to 0}\int_0^\infty f'(t)e^{-st}dt$$

$$\lim_{s\to 0} sF(s) - \cancel{f(0)} = \int_0^\infty f'(t)dt = \lim_{t\to\infty}\int_0^t f'(\tau)d\tau$$

$$= \lim_{t\to\infty} f(t) - \cancel{f(0)}$$

Fundamental Theorem of Calculus

$$\boxed{\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t)}$$

Final Value Theorem

#5 on HW: $\int_0^{\infty} \frac{\sin(t)}{t}\, dt$ let $g(t) = \int_0^{t} \frac{\sin(\tau)}{\tau}\, d\tau$

↳ equivalent to asking $\lim_{t \to \infty} g(t)$